

# Large $N_f$ methods for computing the perturbative structure of deep inelastic scattering \*

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**Abstract.** The large  $N_f$  self consistency method is applied to the computation of perturbative information in the operator product expansion used in deep inelastic scattering. The  $O(1/N_f)$  critical exponents corresponding to the anomalous dimensions of the twist 2 non-singlet and singlet operators are computed analytically as well as the non-singlet structure functions. The results are in agreement with recent explicit perturbative calculations.

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# 1 Introduction

Quantum chromodynamics describes the phenomenology of the strong interactions. Currently there is a need to improve our knowledge of the higher order structure of the field theory beyond earlier one and two loop analysis. Such three loop calculations are performed using efficient computer algebra programmes which handle the tedious algebra of such computations. As these are computer based it is crucial that independent checks are carried out to confirm the final results. We report here on a method which achieves this which is the large  $N_f$  self-consistency approach developed in [1] and applied to four dimensional gauge theories including QED and QCD.[2] Essentially the method formulates perturbation theory in an alternative way. For models which possess parameters additional to the coupling, such as the number of fundamental fields like  $N_f$  quarks in QCD, another expansion parameter exists. If  $N_f$  is large then  $1/N_f$  is small and satisfies the criterion for doing perturbative calculations. Further in this approach[1] one considers the critical behaviour of the field theory in the neighbourhood of the  $d$ -dimensional Gaussian fixed point, where there is a conformal symmetry. As Green's functions obey a simple scaling form one computes the associated critical exponent. This encodes information on the renormalization of the original Green's function. Then the expansion of the exponent in powers of  $\epsilon$ ,  $d = 4 - 2\epsilon$ , gives the coefficients of the corresponding renormalization group function or physical quantity, at that order in  $1/N_f$ . We report here on the application of the technique to the renormalization of the physical operators which arise in the operator product expansion of deep inelastic scattering. In particular, we focus on the twist 2 flavour non-singlet and singlet operators which dominate in most momentum régimes.<sup>3–10</sup> We also discuss the development of the method to compute  $O(1/N_f)$  information on the process dependent moments of the structure functions based on the earlier work of Broadhurst and Kataev[11, 12] inspired by [13].

# 2 Formalism

The twist 2 Wilson operators whose anomalous dimensions we compute in critical exponent form are[3]

$$\mathcal{O}_{NS,a}^{\mu_1 \dots \mu_n} = \frac{1}{2} i^{n-1} \mathcal{S} \bar{q} \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} T_{IJ}^a q - \text{trace terms} \quad (2.1)$$

$$\mathcal{O}_\psi^{\mu_1 \dots \mu_n} = \frac{1}{2} i^{n-1} \mathcal{S} \bar{q} \gamma^{\mu_1} D^{\mu_2} \dots D^{\mu_n} q - \text{trace terms} \quad (2.2)$$

$$\mathcal{O}_G^{\mu_1 \dots \mu_n} = i^{n-2} \mathcal{S} \text{Tr} G^a{}^{\mu_1 \nu} D^{\mu_2} \dots D^{\mu_{n-1}} G_\nu{}^{\mu_n} - \text{trace terms} \quad (2.3)$$

where  $q^{Ii}$  is the quark field,  $1 \leq i \leq N_f$ ,  $1 \leq I \leq N_c$ ,  $n$  is the moment,  $G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c$ ,  $A_\mu^a$  is the gluon field,  $1 \leq a \leq N_c$ , the  $\mathcal{S}$  denotes symmetrization of the Lorentz indices and  $T_{IJ}^a$  are the group generators. We discuss the technique to deduce the anomalous dimensions of (1)-(3) by considering the non-singlet case first.[14] To carry out a perturbative analysis one inserts the operator into some Green's function and determines the pole structure with respect to some regularization. In the critical point approach[1] one deduces critical exponents by using, instead of the usual propagators, the critical ones whose structure is determined solely from scaling arguments and dimensional analysis.[1, 2] So in  $d$ -dimensions the asymptotic scaling forms of the quark and gluon propagators in the critical region,  $k^2 \rightarrow \infty$ , are

$$\tilde{q}(k) \sim \frac{\tilde{A} \not{k}}{(k^2)^{\mu-\alpha}} \quad (2.4)$$

$$\tilde{A}_{\nu\sigma}(k) \sim \frac{\tilde{B}}{(k^2)^{\mu-\beta}} \left[ \eta_{\nu\sigma} - (1-b) \frac{k_\nu k_\sigma}{k^2} \right] \quad (2.5)$$

where  $d = 2\mu$ . The fixed point is defined to be  $g_c$  where  $\beta(g_c) = 0$ ,  $g_c \neq 0$ . From the one loop  $\beta$ -function in  $d$ -dimensions[16, 17, 18]

$$\beta(g) = (d-4)g + \left[ \frac{2}{3}T(R)N_f - \frac{11}{6}C_2(G) \right] g^2 + O(g^3) \quad (2.6)$$

where  $\text{Tr}(T^a T^b) = T(R)\delta^{ab}$ ,  $T^a T^a = C_2(R)I$ ,  $f^{acd} f^{bcd} = C_2(G)\delta^{ab}$  and  $g = (e/2\pi)^2$  is our dimensionless coupling constant. Then from (6) at leading order, as  $N_f \rightarrow \infty$ ,

$$g_c = \frac{3\epsilon}{T(R)N_f} + O\left(\frac{1}{N_f^2}\right) \quad (2.7)$$

The dimension of the fields in (4) and (5) are determined from the fact that the action is dimensionless and their anomalous pieces are defined via

$$\alpha = \mu - 1 + \frac{1}{2}\eta \quad , \quad \beta = 1 - \eta - \chi \quad (2.8)$$

where  $\eta$  is the quark anomalous dimension and  $\chi$  is the anomalous dimension of the quark gluon vertex and each have been calculated at  $O(1/N_f)$  in the Landau gauge.[2] The quantities  $\tilde{A}$  and  $\tilde{B}$  in (4) and (5) are the amplitudes of the fields and  $b$  is the covariant gauge parameter. As (1)-(3) are physical then their anomalous dimensions are gauge independent.[3] So calculating in an arbitrary gauge and observing the cancellation of  $b$  provides a non-trivial check on any calculation, though the work of [3, 4] was carried out in the Feynman gauge.

To proceed in the critical point analysis the regularization is introduced by shifting  $\beta \rightarrow \beta - \Delta$ .[19] Then the residue of the pole in  $\Delta$  when (1) is inserted in a Green's function  $\langle \bar{q}\mathcal{O}_{\text{NS}} q \rangle$  contributes to the renormalization or exponent of the bare operator,  $\gamma_{\mathcal{O}}^{(n)}(g_c)$ . The full gauge independent exponent is given by

$$\gamma_{\text{NS}}^{(n)}(g_c) = \eta + \gamma_{\mathcal{O}}^{(n)}(g_c) \quad (2.9)$$

Then we find the anomalous dimension exponent of (1) is

$$\begin{aligned} \gamma_{\text{NS},1}^{(n)}(g_c) &= \frac{2C_2(R)(\mu-1)^2\eta_1^0}{(2\mu-1)(\mu-2)T(R)} \left[ \frac{(n-1)(2\mu+n-2)}{(\mu+n-1)(\mu+n-2)} \right. \\ &\quad \left. + \frac{2\mu}{(\mu-1)} [\psi(\mu-1+n) - \psi(\mu)] \right] \end{aligned} \quad (2.10)$$

in arbitrary dimensions, where  $\psi(x)$  is the logarithmic derivative of the  $\Gamma$ -function and the subscript 1 denotes the coefficient of  $1/N_f$  in the expansion of  $\gamma_{\text{NS}}^{(n)}(g_c)$ .

There are several checks on (10). First, when  $n = 1$ , the original operator corresponds to a symmetry generator and therefore its anomalous dimension ought to vanish[20]. It is trivial to see that  $\gamma_{\text{NS},1}^{(1)}(g_c) = 0$ . Second, performing the  $\epsilon$ -expansion of (10) and comparing with the explicit perturbative function  $\gamma_{\text{NS}}^{(n)}(g)$ , evaluated at (7), the coefficients agree with the leading order two loop analytic forms given in[3, 4, 5]. Recently information on the *full* 3-loop structure has been provided.[21] That calculation made extensive use of a computer algebra programme to the extent that  $\gamma_{\text{NS}}^{(n)}(g)$  is known at 3-loops for the even moments up to  $n = 10$ . Expanding (10) to  $O(\epsilon^3)$  one obtains the analytic expression

$$\begin{aligned} a_3^{\text{NS}} &= \frac{2}{9}S_3(n) - \frac{10}{27}S_2(n) - \frac{2}{27}S_1(n) + \frac{17}{72} \\ &\quad - \frac{[12n^4 + 2n^3 - 12n^2 - 2n + 3]}{27n^3(n+1)^3} \end{aligned} \quad (2.11)$$

for the 3-loop coefficient at  $O(1/N_f)$ , where  $S_l(n) = \sum_{i=1}^n 1/i^l$ . Substituting for the various  $n$  we record that (11) is in exact agreement with[21]. Analytic expressions at  $O(1/N_f)$  can be obtained for the higher order coefficients.[14]

### 3 Singlet Anomalous Dimensions

We now turn to the singlet case. As (2) and (3) have the same dimensions they mix under renormalization which complicates the computation of their anomalous dimensions. By contrast to (1) one has a matrix of anomalous dimensions,  $\gamma_{ij}^{(n)}(g)$ .[3] Its eigenvalues correspond to the anomalous dimensions of two independent combinations of the original (bare) operators which are of physical interest. Perturbatively  $\gamma_{ij}^{(n)}(g)$  is determined in the same way as  $\gamma_{\text{NS}}^{(n)}(g)$  but the operators are also included in a gluon 2-point function  $\langle AOA \rangle$  and it, and therefore its eigenvalues  $\gamma_{\pm}^{(n)}(g)$ , have been computed to two loops.<sup>7–10</sup> To construct the corresponding anomalous dimension exponents, one analyses the leading order large  $N_f$  graphs at criticality as discussed before but now determines the matrix of critical exponents. The graphs for these have been given elsewhere[7] but we note that there are several two loop graphs which contribute at leading order in  $1/N_f$ . We find the eigenvalues of  $\gamma_{ij}^{(n)}(g)$  at criticality for  $n$  even, at leading order, are

$$\gamma_+^{(n)}(g_c) = -2(\mu - 2) \quad (3.1)$$

$$\begin{aligned} \gamma_-^{(n)}(g_c) = & \frac{C_2(R)\eta_1^0}{(2\mu - 1)(\mu - 2)T(R)N_f} \left[ \frac{2(\mu - 1)^2(n - 1)(2\mu + n - 2)}{(\mu + n - 1)(\mu + n - 2)} \right. \\ & + 4\mu(\mu - 1)[\psi(\mu - 1 + n) - \psi(\mu)] \\ & - \frac{\mu(\mu - 1)\Gamma(n - 1)\Gamma(2\mu)}{(\mu + n - 1)(\mu + n - 2)\Gamma(2\mu - 1 + n)} \\ & \times [(n(n - 1) + 2(\mu - 1 + n))^2 \\ & \left. + 2(\mu - 2)(n(n - 1)(2\mu - 3 + 2n) + 2(\mu - 1 + n))] \right] \end{aligned} \quad (3.2)$$

In the critical point approach (12) and (13) follow naturally without diagonalization as  $\gamma_{ij}^{(n)}(g_c)$  is triangular at leading order. Eq. (12) represents the anomalous dimension of the operator which has (3) as its dominant contribution in the linear combination we mentioned, whilst (13) corresponds to (2) being dominant. The  $N_f$  dependence of each term is different due to the particular way in which quark loops occur at leading order in the Green's functions. Specifically the  $\epsilon$ -expansion of (12) involves only a one loop term and no subsequent terms, consistent with the perturbative analysis. (We also note that recently the renormalization of the gluonic sector has been re-examined to understand the role infrared divergences play.[22])

There are several checks on (13). When  $n = 2$ , the operator corresponds to the energy momentum tensor and since it is conserved its anomalous dimension vanishes.[20] Clearly,  $\gamma_-^{(2)}(g_c) = 0$ . Secondly, using the explicit 2-loop matrix,  $\gamma_{ij}^{(n)}(g)$ , and extracting both eigenvalues as a perturbative expansion in  $g$ , the coefficients of (13) to  $O(\epsilon^2)$  agree exactly.<sup>7–10</sup> We have also checked the cancellation of the gauge parameter. For comparison sake we record that the three loop leading order  $1/N_f$  coefficient of  $\gamma_-^{(n)}(g)$  is

$$\begin{aligned} a_3^\psi = & \frac{2}{9}S_3(n) - \frac{10}{27}S_2(n) + \frac{17}{72} - \frac{2(n^2 + n + 2)^2[S_2(n) + S_1^2(n)]}{3n^2(n + 2)(n + 1)^2(n - 1)} \\ & - 2S_1(n)[n^9 + 6n^8 - 36n^7 - 216n^6 - 552n^5 - 810n^4 - 811n^3 \\ & - 690n^2 - 132n + 72]/[27(n + 2)^2(n + 1)^3(n - 1)n^3] \\ & - [100n^{10} + 682n^9 + 2079n^8 + 3377n^7 + 3389n^6 + 3545n^5 + 3130n^4 \\ & + 118n^3 - 940n^2 - 72n + 144]/[27(n + 2)^3(n + 1)^4n^4(n - 1)] \end{aligned} \quad (3.3)$$

## 4 Structure Function Moments

A second ingredient in the Wilson expansion is the determination of the process dependent Wilson coefficients,  $C^i(q^2/m^2, g)$ ,  $i = \text{NS}, \psi$  or  $G$ , whose momentum evolution is controlled by the anomalous dimensions. As these have also been computed at  $O(g^3)$  and to  $n = 10$  in [21] it is equally important to have an efficient method of calculation in large  $N_f$ . By contrast with (10,12,13), which relate to the divergent part of Green's functions, the  $C^i$ 's are determined from the *finite* part of the amplitude after renormalization. The most efficient way to achieve this is based on [11, 12] where one computes a minimal set of one or two loop graphs. In that approach the relevant diagram is calculated in strictly four dimensions but where the gluon line has an exponent of  $1 + \delta$  instead of 1. Multiplying the resultant expression, which will be a function of  $\delta$ , by  $e^{5\delta/3}$  the coefficients of its Taylor series in  $\delta$  correspond to the perturbative coefficients of  $C^i$  at  $O(1/N_f)$ . This exponential factor is necessary to ensure results are in the  $\overline{\text{MS}}$  scheme, and the 5/3 arises from the finite part of the charge renormalization. (We note that we have developed this independently of a similar approach in [23] for a different problem.) We have applied this method to the determination of the simplest process ie the non-singlet longitudinal Wilson coefficient in the Bjorken limit. Including the crossed amplitude we find, at the  $L$ th loop,

$$C_{\text{long}}^{\text{NS}}(1, g) = \frac{d^L}{d\delta^L} \left[ \frac{8C_2(R)e^{5\delta/3}\Gamma(n + \delta)g}{(2 - \delta)(1 - \delta)(n + 1 + \delta)\Gamma(n)\Gamma(1 + \delta)x^n} \right]_{\delta = \frac{4}{3}N_f T(R)g} \quad (4.1)$$

We recall that the coupling constant expansion of  $C_{\text{long}}^{\text{NS}}(1, g)$  is  $O(g)$  and not  $O(1)$ . Expanding in powers of  $\delta$  the coefficients agree with the 2-loop results of [7, 24, 25] for all  $n$ . We note that the three loop coefficient at  $O(1/N_f)$  is

$$\begin{aligned} a_{\text{long},3}^{\text{NS}} &= \frac{16C_2(R)T^2(R)}{9(n+1)} \left[ \frac{203}{18} - S_2(n) + S_1^2(n) - \frac{19(2n+1)}{3n(n+1)} \right. \\ &\quad \left. + 2S_1(n) \left( \frac{19}{6} - \frac{1}{n} - \frac{1}{n+1} \right) + \frac{2}{n^2} + \frac{2}{(n+1)^2} + \frac{2}{n(n+1)} \right] \end{aligned} \quad (4.2)$$

which agrees with the even moments of [21] up to  $n = 10$ .

## 5 Conclusions

We have presented the latest results of applying the large  $N_f$  self-consistency programme to the operator product expansion of deep inelastic scattering. Analytic results have been obtained for the higher order structure of the anomalous dimensions of the physical operators and the moments of the structure functions at  $O(1/N_f)$ . An important feature of our results is that we have given an insight into the (complicated) analytic structure at three and higher loops as a function of  $n$ , primarily by exploiting the conformal symmetry of the  $d$ -dimensional fixed point. This is a first step in gaining an insight into the subleading  $O(1/N_f^2)$  coefficients in relation to determining the  $O(g^3)$  Altarelli-Parisi splitting functions. Further, we have indicated how to deduce useful information compactly and efficiently, for the moments of Wilson coefficients in  $1/N_f$ . Although we have focussed on the simplest case, the longitudinal part of the non-singlet amplitude, it ought to be possible to adapt the approach to determine information on the singlet structure.

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